Tesla coil theory

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1 Abstract

This article makes a simple theoretical approach to the tesla coil. Theoretical means that we're going to use physics and mathematics, simple means that we restrict ourselves so some special cases.

We'll treat the tesla coil as two inductively coupled, damped resonant circuits. The corresponding system of linear differential equations will be solved for the special case of no damping.

2 Model and differential equations

Figure 1 shows the model circuit we're going to examine. Note that C_2 includes the self-capacitance of the the secondary coil as well any extra capacity e.g. of the top terminal. R_1 and R_2 represent unavoidable series resistances.

L_1, C_1, R_1 and L_2, C_2, R_2 form two damped resonant circuits, that are coupled by the mutual inductance M. According to Kirchhoff's laws, the currents I_1 and I_2 obey the following system of integro-differential euqations:

$$\begin{aligned} R_1 I_1 \;+\; \frac{1}{C_1}\int I_1\,dt \;+\; L_1\dot{I}_1 \;+\; M\dot{I}_2 \;&= 0 \\ R_2 I_2 \;+\; \frac{1}{C_2}\int I_2\,dt \;+\; L_2\dot{I}_2 \;+\; M\dot{I}_1 \;&= 0 \end{aligned} \tag{1}$$

We are interested more in voltages than in currents, so we look at the voltage across the capacitors

$$U_j = \frac{1}{C_j}\int I_j \qquad I_j = C_j\dot{U}_j \qquad j = 1,2 \tag{2}$$

Now (1) can be expressed in terms of U_j:

$$\begin{aligned} R_1 C_1\dot{U}_1 \;+\; U_1 \;+\; L_1 C_1\ddot{U}_1 \;+\; M C_2\ddot{U}_2 \;&= 0 \\ R_2 C_2\dot{U}_2 \;+\; U_2 \;+\; L_2 C_2\ddot{U}_2 \;+\; M C_1\ddot{U}_1 \;&= 0 \end{aligned} \tag{3}$$

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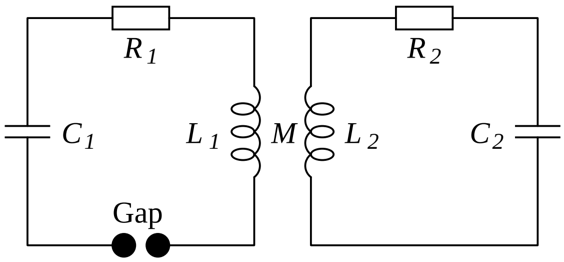

Figure 1: tesla coil model circuit

Substituing the Ansatz

$$U_j(t) = \hat{U}_j e^{i\omega t} \tag{4}$$

into the differential equations (3), we get (writing U instead of $\hat{U}$ again)

$$\begin{aligned} K_1 U_1 \; - \; \omega^2 M C_2 U_2 \; &= 0 \\ K_2 U_2 \; - \; \omega^2 M C_1 U_1 \; &= 0 \end{aligned} \tag{5}$$

where

$$K_j = i\omega R_j C_j + 1 - \omega^2 L_j C_j .$$

This is an ordinary system of linear equations for U_1 and U_2. From the condition that it must have non-trivial solutions (i.e. not $U_1 = U_2 = 0$), we will determine the possible values of ω. A non-trivial solution exists if and only if the determinant

$$K_1 K_2 - \omega^2 M^2 C_1 C_2 = 0. \tag{6}$$

This polynomial equation of 4th degree for ω has four complex solutions. In case they are all different, the system is called non-degenerate, and the functions $U_j e^{i\omega t}$ are then a basis of the space of solutions of (3). This means that any solution of (3) is a linear combination of such functions.

3 Solving the simplified model

As a first approximation, we assume that the damping in both circuits is small enough to be neglected, i.e. $R_1 = R_2 = 0$. Note that this is not necessarily a good approximation, as a tesla coil secondary usually has several hundred meters of thin wire.

The system (5) has a non-trivial solution if and only if the determinant is zero. Setting $R_1 = 0$, $R_2 = 0$ and introducing the abbreviations

$$\frac{M^2}{L_1 L_2} = k^2 \quad \frac{1}{L_1 C_1} = \omega_1^2 \quad \frac{1}{L_2 C_2} = \omega_2^2$$

(6) is simplified to

$$(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) - \omega^4 k^2 = 0. \tag{7}$$

This quadratic equation in ω^2 has two solutions

$$\omega_{\mathrm{I,II}}^2 = \frac{1}{2(1-k^2)} \left(\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2 \omega_2^2 (1-k^2)} \right) \tag{8}$$

Any solution of the simplified system (1) is a linear combination of the four eigensolutions:

$$U_j(t) = A_j e^{i\omega_{\mathrm{I}} t} + B_j e^{-i\omega_I t} + C_j e^{i\omega_{\mathrm{II}} t} + D_j e^{-i\omega_{\mathrm{II}} t} \tag{9}$$

Each pair of coefficients $A_j, \ldots, D_j$ must suffice (5), which leads to

$$X_2 = \frac{1}{k}\sqrt{\frac{L_2}{L_1}}\frac{\omega_2^2}{\omega_1^2}\left(\frac{\omega_1^2}{\omega^2} - 1\right) X_1 \equiv S(\omega) X_1, \tag{10}$$

where X is any coefficient and $\omega = \omega_{\mathrm{I}}, \omega_{\mathrm{II}}$ respectively as in (9). The remaining four coefficients $A_1, \ldots, D_1$ are determined by the initial state of the system, which in our case is

$$\begin{array}{ll} \dot{U}_1(0) = 0 & U_1(0) = U_p \\ \dot{U}_2(0) = 0 & U_2(0) = 0 \end{array} \tag{11}$$

Substituting (9) for U_j and making use of (10) leads to a system of equations for $A_1, \ldots, D_1$. A simple but somewhat lengthy calculation leads to

$$\begin{array}{ll} A_1 = B_1 & A_1 = \dfrac{U_p}{2}\dfrac{S(\omega_{\mathrm{II}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \\[2ex] C_1 = D_1 & C_1 = -\dfrac{U_p}{2}\dfrac{S(\omega_{\mathrm{I}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \end{array} \tag{12}$$

So what do these admittedly not friendly equations tell us? Let's look at the special case

$$\omega_1 = \omega_2 \equiv \omega,$$

i.e., primary and secondary circuit are in tune. In this case, we find

$$\begin{aligned} \omega_{\mathrm{I,II}} &= \frac{\omega}{\sqrt{1 \pm k^2}} \\ S(\omega_{\mathrm{I,II}}) &= \pm\sqrt{\frac{L_2}{L_1}} \\ A_1 = B_1 &= \frac{U_p}{4} \\ C_1 = D_1 &= \frac{U_p}{4} \\ A_2 = B_2 &= \sqrt{\frac{L_2}{L_1}}\frac{U_p}{4} \\ C_2 = D_2 &= -\sqrt{\frac{L_2}{L_1}}\frac{U_p}{4} \end{aligned}$$

For the primary and secondary voltage we find (using $2\cos(\alpha) = e^{i\alpha} + e^{-i\alpha}$)

$$\begin{aligned} U_1 &= \frac{U_p}{2}\left(\cos(\omega_{\mathrm{I}} t) + \cos(\omega_{\mathrm{II}} t)\right), \\ U_1 &= \sqrt{\frac{L_2}{L_1}}\frac{U_p}{2}\left(\cos(\omega_{\mathrm{I}} t) - \cos(\omega_{\mathrm{II}} t)\right), \end{aligned} \tag{13}$$

or, using an addition theorem,

$$
\begin{aligned}
U_1 &= U_p \cos\left(\frac{\omega_{\mathrm{I}} + \omega_{\mathrm{II}}}{2} t\right) \cos\left(\frac{\omega_{\mathrm{I}} - \omega_{\mathrm{II}}}{2} t\right), \\
U_2 &= \sqrt{\frac{L_2}{L_1}} U_p \sin\left(\frac{\omega_{\mathrm{I}} + \omega_{\mathrm{II}}}{2} t\right) \sin\left(\frac{\omega_{\mathrm{I}} - \omega_{\mathrm{II}}}{2} t\right).
\end{aligned}
\tag{14}
$$

Now lets return to the more general case $\omega_1 \neq \omega_2$, but still $R_1 = R_2 = 0$. We have already determined the coefficients $A_1, \ldots, D_1$, see (12). From (10) we calculate

$$
\begin{aligned}
A_2 = B_2 = S(\omega_{\mathrm{I}}) A_1 &= \frac{U_p}{2} \frac{S(\omega_{\mathrm{I}}) S(\omega_{\mathrm{II}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \\
C_2 = D_2 = S(\omega_{\mathrm{II}}) C_1 &= -\frac{U_p}{2} \frac{S(\omega_{\mathrm{I}}) S(\omega_{\mathrm{II}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})}
\end{aligned}
\tag{15}
$$

and find for the primary and secondary voltage (using the same addition theorems as above)

$$
\begin{aligned}
U_1 &= \frac{U_p}{2} \left(\frac{S(\omega_{\mathrm{II}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \cos(\omega_{\mathrm{I}} t) - \frac{S(\omega_{\mathrm{I}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \cos(\omega_{\mathrm{II}} t) \right) \\
U_2 &= -2 U_p \frac{S(\omega_{\mathrm{I}}) S(\omega_{\mathrm{II}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \sin\left(\frac{\omega_{\mathrm{I}} + \omega_{\mathrm{II}}}{2} t\right) \sin\left(\frac{\omega_{\mathrm{I}} - \omega_{\mathrm{II}}}{2} t\right).
\end{aligned}
\tag{16}
$$

The primary voltage is no longer a simple beat, but the secondary voltage form is the same as in the tuned case.

We want to have a look at the secondary peak voltage as depending on the "mistune". We define the transmission ratio

$$
r = 2 \left| \frac{S(\omega_{\mathrm{I}}) S(\omega_{\mathrm{II}})}{S(\omega_{\mathrm{II}}) - S(\omega_{\mathrm{I}})} \right|. \tag{17}
$$

The highest value U_2 possibly can take is rU_p. Inserting the definition of $S(\omega)$ from (10) and simplifying a bit, we get

$$
r = 2 \left| \frac{1}{k} \sqrt{\frac{L_2}{L_1}} \frac{\omega_2^2}{\omega_1^2} \frac{\omega_1^4 - \omega_1^2(\omega_{\mathrm{I}}^2 + \omega_{\mathrm{II}}^2) + \omega_{\mathrm{I}}^2 \omega_{\mathrm{II}}^2}{\omega_1^2(\omega_{\mathrm{I}} - \omega_{\mathrm{II}})} \right|. \tag{18}
$$

In order to eliminate the remaining $\omega_{\mathrm{I}}, \omega_{\mathrm{II}}$ we make use of the fact that they solve (3) and of Vieta's theorem

$$
\begin{aligned}
\omega_{\mathrm{I}}^2 \omega_{\mathrm{II}}^2 &= \frac{\omega_1^2 \omega_2^2}{1 - k^2} \\
\omega_{\mathrm{I}}^2 + \omega_{\mathrm{II}}^2 &= \frac{\omega_1^2 + \omega_2^2}{1 - k^2},
\end{aligned}
$$

as well as the explicit solution (8)

$$
\omega_{\mathrm{I}}^2 - \omega_{\mathrm{II}}^2 = \frac{1}{1 - k^2} \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2 \omega_2^2 (1 - k^2)}.
$$

Substituting this into (18) and defining $x = \frac{\omega_1}{\omega_2} = \sqrt{\frac{L_2}{L_1}\frac{C_2}{C_1}}$ we finally get

$$r = \sqrt{\frac{C_1}{C_2}} \frac{2k}{\sqrt{\left(x + \frac{1}{x}\right)^2 - 4(1 - k^2)}}. \tag{19}$$

Obviously r takes on its maximum for $x = 1$ for fixed capacities C_1 and C_2 (this is realistic as tesla coils are usually tuned by changing the primary inductance). In this case it simplifies to

$$r_{\max} = \sqrt{\frac{C_1}{C_2}} = \sqrt{\frac{L_2}{L_1}}. \tag{20}$$

References

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